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The relaxation of stress in a ν-fluid with reference to the decay of homogeneous turbulence

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It was suggested by Proudman (1970) that many of the phenomena of turbulence at high Reynolds number could be modelled by a suitably chosen member of a class of non-Newtonian fluids, ν-fluids, all of whose properties depend only on a single dimensional constant with the dimensions of viscosity. This paper investigates the relaxation of homogeneous stress in a doubly degenerate third-order ν-fluid (which is the simplest member of the class that can possibly be used to model turbulence) in the limit $\nu \rightarrow 0$.

The equation which governs the stress tensor S is of the form

$$AS\ddot{S} + B\dot{S}^2 = 0,$$

where A and B are isotropic tensor constants of the fluid; its differential structure can be of four distinct types. A list of properties required of the solutions of the equation is set out, and it is shown that only one of the four types has all the properties demanded of it. The behaviour of solutions of this equation is found to be consistent with the theoretical and experimental results on the decay of homogeneous turbulence.

1. Introduction

In his paper on the motion of ν-fluids, Proudman (1970) shows that some properties of turbulence can be described by a third-order ν-fluid if an appropriate choice is made for the constants which appear in the governing equation for the stress tensor S. At present, little is known about the behaviour of solutions of equations of this type, but a description of turbulent phenomena by relatively simple equations such as these is clearly of such great potential value that it is worth investigating them further. The result of such an investigation will either be a demonstration that some important known properties of turbulence cannot be described in this way, or else it will show that some relation or set of relations must be satisfied by the constants in the equation for the stress if these properties are to be described adequately.

In view of the large number of constants in the governing equation of a ν-fluid it is preferable to consider some special phenomena which allow an investigation of their dependence on a smaller group of constants. Thus the phenomenon investigated here is that of relaxation of homogeneous stress in a stationary third-order ν-fluid, and the conditions under which it can be used to model the decay of homogeneous turbulence at infinite Reynolds number.

The properties of homogeneous turbulence which must be shared by S if $-S$ is to describe the Reynolds stress tensor $\{-\overline{u_i'u_j'}\}$ for this problem, where $\mathbf{u}'$ is the turbulent velocity fluctuation, and which will be required of solutions of any equation for S studied in this paper as a model for turbulence, are as follows.

(i) S must be real and positive semi-definite for all values of the time t at and after the initial instant. This condition is an immediate consequence of the definition of the Reynolds stress tensor.

(ii) S must satisfy the thermodynamic condition $\dot{\sigma} < 0$, where $\sigma = \mathrm{tr}\,(S)$.

(iii) The experimental result that the energy, and hence σ, decays to zero a long time after the relaxation started. The form of the decay can be approximated well by a power law for isotropic and slightly anisotropic homogeneous turbulence and, in experiments in wind tunnels, is independent of a Reynolds number based on grid spacing and mean velocity when this number is very large.

(iv) The experimental result that, if the stress tensor in homogeneous turbulence is initially anisotropic and has a high Reynolds number, then it has a tendency towards isotropy.

Proudman defines an nth-order ν-fluid as one in which S, at a point where the velocity is $\mathbf{u}$, is governed by an equation of the form

$$\frac{D^nS}{Dt^n} = f\left(\mathbf{u}, \frac{D\mathbf{u}}{Dt}, \ldots, \frac{D^n\mathbf{u}}{Dt^n}, S, \frac{DS}{Dt}, \ldots, \frac{D^{n-1}S}{Dt^{n-1}}, \nu\right),$$

where f is a function which is regular at the origin of the multi-dimensional space of all the arguments shown and all the space derivatives of any order. He concludes that the principle of linear material indifference must apply to the fluid if it is to model turbulence and hence that

$$\mathbf{u}, \frac{D\mathbf{u}}{Dt}, \ldots, \frac{D^n\mathbf{u}}{Dt^n}$$

must not occur explicitly in the function, although their space derivatives may. Further, he requires $S = 0$ to be a possible solution of the equation.

These conditions lead him to a family of fluids of different orders; the equations which govern first-, second- and third-order fluids, respectively, reduce to the forms

$$\dot{S} = \frac{1}{\nu} p_1^1 S^2 + O(\nu^0) \quad (n = 1),$$

$$\ddot{S} = \frac{1}{\nu^2} p_1^2 S^3 + \frac{1}{\nu} p_2^2 S\dot{S} + O(\nu^0) \quad (n = 2),$$

$$\dddot{S} = \frac{1}{\nu^2} p_1^3 S^4 + \frac{1}{\nu^2} p_2^3 S^2\dot{S} + \frac{1}{\nu}(p_4^3 S\ddot{S} + p_5^3 \dot{S}^2) + O(\nu^0) \quad (n = 3)$$

for the relaxation of homogeneous stress in the absence of a velocity field $\mathbf{u}$; the quantities p_β^α are non-dimensional isotropic tensor constants of the fluid. If the fluid is to model the decay of homogeneous turbulence at high Reynolds number, the equation must possess a non-trivial solution in the limit as $\nu \to 0$. Thus for a first-order fluid the only possible form of equation for S in the limit is

$$p_1^1 S^2 = 0.$$

Since the equations are universal, this can only be a condition of isotropy and so, at large Reynolds number, a first-order ν-fluid must have an approximately isotropic stress system; departures from this must decay through transients of very short duration. This behaviour is quite unacceptable in a model which attempts to describe the general behaviour of homogeneous but not necessarily isotropic turbulence, and so a first-order fluid must be rejected. For the same reason a second-order fluid in which p_1^2 is non-zero must be rejected. However, if $p_1^2 = 0$, the limit becomes

$$p_2^2 S\dot{S} = 0;$$

it is clear that this equation cannot adequately describe the decay of turbulent energy and so a second-order fluid must be rejected.

For a third-order fluid the same arguments show that the fluid must be doubly degenerate; so

$$p_1^3 = p_2^3 = 0,$$

giving a constitutive relation in the limit as $\nu \to 0$ of the form

$$a_1 \ddot{S}\sigma + a_2 S\ddot{\sigma} + a_3(S\ddot{S} + \ddot{S}S) + a_4 I\sigma\ddot{\sigma} + a_5 I \operatorname{tr}(S\ddot{S}) + a_6 \dot{\sigma}\dot{S} \\ + a_7 \dot{S}^2 + a_8 I\dot{\sigma}^2 + a_9 I \operatorname{tr}(\dot{S}^2) = 0. \quad (1.1)$$

For a suitable choice of the constants a_1–a_9, this equation gives a power-law decay for isotropic stress. It is thus clear that a doubly degenerate third-order ν-fluid is the simplest ν-fluid which can possibly be used to model the phenomena of turbulence. The aim here is to restrict the values of these constants in such a way that the solutions of (1.1) agree satisfactorily with all the properties of the turbulent stress tensor listed above.

In this connexion we may note that certain relations between the constants lead to degeneracies in the differential structure of (1.1) over and above those considered by Proudman, which were essentially concerned with the limiting process $\nu \to 0$. Thus doubly degenerate fluids may be divided into four types.

Type I. Equation (1.1) determines all components of $\ddot{S}$ for all but special values of S.

Type II. Equation (1.1) determines more than one linear combination of the components of $\ddot{S}$ for all but special values of S, but does not determine them all.

Type III. Equation (1.1) determines only one linear combination of the components of $\ddot{S}$ for all but special values of S.

Type IV. Equation (1.1) does not involve second derivatives.

If

$$a_1 = a_2 = a_3 = 0 \quad \text{or} \quad a_1 = a_3 = a_5 = 0$$

the fluid is of type III, and if

$$a_1 = a_2 = a_3 = a_4 = a_5 = 0$$

it is of type IV. All other fluids are either of type I or type II.

The last of these four types can be rejected at once for the same reasons that led to the rejection of first- and second-order ν-fluids. It is possible to investigate the remaining three types for arbitrary initial conditions and for entirely arbitrary forms of the equation, but attention will be restricted to those forms which possess the following properties.

Restriction 1. The stress tensor remains positive definite if it is initially so.

Restriction 2. The trace σ of the stress tensor is determined by the equation and is monotonically decreasing whenever S is positive definite.

Sections 2 and 3 of this paper are devoted to the study of fluids whose stress system is governed by (1.1) and which must have these two properties. The principal result of § 2 is that the asymptotic stress in fluids of types I and II is not isotropic for arbitrary initial conditions, and the principal result of § 3 is that the asymptotic stress in fluids of type III is always isotropic (and previously well-behaved) for arbitrary initial conditions provided that certain inequalities in the fluid constants are satisfied. These results will be obtained for the initial conditions appropriate to the differential type of the equation being studied, consistent only with the additional requirements $\boldsymbol{\lambda}'S\boldsymbol{\lambda} \geqslant 0$ for all real vectors $\boldsymbol{\lambda}$ and $\dot{\sigma} < 0$ at the initial instant.

The second of these two results implies that a fluid of type III is acceptable as a model for turbulence in the sense that, with appropriate conditions on the constants in the equation, all the properties of homogeneous turbulence listed above are satisfied.

The first of these results, however, has the consequence that fluids of types I and II can only be accepted as models of turbulence if constraints on the initial conditions are imposed in addition to those required by the differential structure of the governing equation; thus for fluids of type I the initial conditions required by the differential structure of the equation are values of S and $\dot{S}$ with the given value of S positive definite and that of $\dot{\sigma}$ negative for consistency with restrictions 1 and 2. There is no evidence to suggest that there are any further restrictions on S or $\dot{S}$ if S is to model the decay of stress in homogeneous turbulence; consequently, at infinite Reynolds number, further constraints deduced from the equation itself (if a fluid of type I or II is to be accepted) seriously undermine Proudman's approach, which is based heavily on universality. (The position at finite Reynolds number may be different; his approach is asymptotic as $\nu \rightarrow 0$ and it may be necessary to restrict initial conditions to be close in some sense to those appropriate to $\nu = 0$.)

In the light of these remarks and the results of §§ 2 and 3, it follows that the only fluids whose stress systems are governed by (1.1) which can serve as models for turbulence are those of type III, and they are studied further in § 4. It will be shown there that the exact solution of the equation which governs the stress is

$$S = \frac{S_0 - \frac{1}{3}\sigma_0 I}{(1+t/t_0)^{1+n}} + \frac{\frac{1}{3}\sigma_0 I}{(1+t/t_0)^{1+r}},$$

where S_0 and σ_0 are the values of S and σ at $t = 0$ and

$$t_0 = (1+r)\,\sigma_0/|\dot{\sigma}_0|,$$

where $\dot{\sigma}_0$ is the initial value of $\dot{\sigma}$. The numbers r and n are functions of the constants of the equation. This solution is compared with the results of experiments on the decay of homogeneous turbulence at large Reynolds numbers.

2. Fluids of types I and II

The principal aim of this section is to show that S does not necessarily tend to the isotropic state for every possible initial condition when the fluid is of type I or type II and satisfies the two additional restrictions 1 and 2. Since the result is true within the restricted class of axisymmetric forms for S, it is sufficient to consider only this case. It is possible to establish further constraints on the constants of the fluid if the two restrictions are to be satisfied for general forms of S, and this is done.

A large class of fluids of both types are those in which σ is not determined when S is isotropic, and such fluids are not discussed here since they violate restriction 1. The axisymmetric relaxation of all further fluids of these two types may be discussed by the same methods and it is unnecessary to differentiate between them.

To study axisymmetric relaxation, write

$$S_{ij} = \tfrac{1}{3}(\sigma - g)\,\delta_{ij} + g\lambda_i\lambda_j, \quad \text{where} \quad \sum_{i=1}^{3} \lambda_i^2 = 1;$$

it follows that S is a solution of (1.1) provided that

$$(\tfrac{1}{3}a_1 + \tfrac{1}{3}a_2 + \tfrac{2}{9}a_3 + a_4 + \tfrac{1}{3}a_5)\,\sigma\ddot{\sigma} + \tfrac{2}{3}(\tfrac{2}{3}a_3 + a_5)\,g\ddot{g}$$
$$+ (\tfrac{1}{3}a_6 + \tfrac{1}{9}a_7 + a_8 + \tfrac{1}{3}a_9)\,\dot{\sigma}^2 + \tfrac{1}{3}(\tfrac{2}{3}a_7 + 2a_9)\,\dot{g}^2 = 0 \qquad (2.1)$$

and
$$(a_1 + \tfrac{2}{3}a_3)\,\sigma\ddot{g} + \tfrac{2}{3}a_3 g\ddot{g} + (a_2 + \tfrac{2}{3}a_3)\,g\ddot{\sigma} + (a_6 + \tfrac{2}{3}a_7)\,\dot{g}\dot{\sigma} + \tfrac{1}{3}a_7\dot{g}^2 = 0. \qquad (2.2)$$

These equations can possess solutions for which g vanishes at all times, and this corresponds to the decay of isotropic stress. The solution is then of the form

$$\sigma = R/(t-t_0)^{1+r},$$

where R and t_0 are fixed by the initial conditions, but r is determined by the coefficient in (2.1). If the stress is to decay to zero as t tends to infinity, r must be greater than -1. The coefficients of the first and third terms of the equation must therefore both be non-zero and in the ratio

$$1 : -\frac{2+r}{1+r},$$

with $r > -1$.

Now suppose that the equations are solved with initial conditions very close to isotropy; the solution can then be expressed in the form

$$R/(t-t_0)^{1+r} + \epsilon R_1(t) + O(\epsilon^2),$$

where ϵ is small compared with R. The function $R_1(t)$ is of order 1 and satisfies the equation

$$(t-t_0)^2\,(a_1 + \tfrac{2}{3}a_3)\,\ddot{R}_1 - (t-t_0)\,(1+r)\,(a_6 + \tfrac{2}{3}a_7)\,\dot{R}_1 + (1+r)\,(2+r)\,(a_2 + \tfrac{2}{3}a_3)\,R_1 = 0.$$

This equation has a general solution for R_1 of the form

$$\frac{a}{(t-t_0)^{1+m}} + \frac{b}{(t-t_0)^{1+n}},$$

where a and b are determined by the initial conditions but m and n are given in terms of a_1, a_2, a_3, a_6 and a_7. We may note that if the solution is to tend to isotropy then n and m should both be greater than r. This also gives the asymptotic form of decay for large values of t for those solutions which tend to isotropy.

The information obtained so far can now be expressed in terms of m, n and r by means of the ratios

$$\tfrac{1}{3}a_1+\tfrac{1}{3}a_2+\tfrac{2}{9}a_3+a_4+\tfrac{1}{3}a_5:\tfrac{4}{9}a_3+\tfrac{2}{3}a_5:\tfrac{2}{9}a_7+\tfrac{2}{3}a_9:\tfrac{1}{3}a_6+\tfrac{1}{9}a_7+a_8+\tfrac{1}{3}a_9 = 1:\alpha:\beta:-\frac{2+r}{1+r}$$

and

$$a_2+\tfrac{2}{3}a_3:a_1+\tfrac{2}{3}a_3:\tfrac{2}{3}a_3:a_6+\tfrac{2}{3}a_7:\tfrac{1}{3}a_7 = \frac{(1+n)(1+m)}{(1+r)(2+r)}:1:\gamma:-\frac{n+m+3}{1+r}:\delta,$$

where α, β, γ and δ are parameters.

Equations (2.1) and (2.2) can now be solved for $\ddot{\sigma}$ and $\ddot{g}$:

$$\left\{\sigma^2+\gamma\sigma g-\alpha g^2\frac{(1+n)(1+m)}{(1+r)(2+r)}\right\}\ddot{\sigma} = \dot{g}^2(\alpha\delta g-\beta\sigma-\beta\gamma g) -\alpha g\dot{g}\dot{\sigma}\frac{n+m+3}{1+r}+\frac{2+r}{1+r}(\sigma+\gamma g)\dot{\sigma}^2 \qquad (2.3)$$

and

$$\left\{\sigma^2+\gamma\sigma g-\alpha g^2\frac{(1+n)(1+m)}{(1+r)(2+r)}\right\}\ddot{g} = g\frac{(1+n)(1+m)}{(1+r)(2+r)}\left(\beta\dot{g}^2-\frac{2+r}{1+r}\dot{\sigma}^2\right) +\sigma\dot{\sigma}\dot{g}\frac{n+m+3}{1+r}-\delta\sigma\dot{g}^2. \qquad (2.4)$$

The condition that the principal stresses must be non-negative gives further information about γ and α. Suppose that the initial conditions are such that σ is positive and $\dot{\sigma}$ is negative; then so long as $\dot{\sigma}$ is not zero, $\dot{g}$ can be specified arbitrarily and g then need only satisfy the condition that the principal stresses are non-negative. In particular, $\dot{\sigma}$ can be specified so that the fluid is initially impelled away from isotropy; in other words, the lesser of the two principal stresses can be made to decrease at an arbitrarily high rate. It is clear that such conditions could not be applied for physical reasons if this principal stress were zero, but they are otherwise acceptable and imply that a principal stress very small but positive would become negative after a finite time if the initial tendency away from isotropy were made large enough. The only conditions under which this would not apply are when the vanishing of either principal stress gives a singular point of the equation. This requires that the factor on the left-hand sides of (2.3) and (2.4) is $(\sigma-g)(\sigma+2g)$. Consequently

$$\gamma = 1, \quad \alpha = 2(1+r)(2+r)/(1+n)(1+m).$$

Now, in order to study the behaviour of σ, consider initial conditions in which $\dot{\sigma}$ is arbitrarily small, $\dot{g}$ is non-zero and g does not take either of the possible extreme values $-\tfrac{1}{2}\sigma$ or σ. The sign of $\ddot{\sigma}$ is then the same as the sign of

$$(\alpha\delta-\gamma\beta)g-\beta\sigma.$$

Since $\dot{\sigma}$ cannot be positive, this factor must either be identically zero or else must be negative for all acceptable values of g with, again, the possible exception of the extreme values. If this factor is identically zero then

$$\beta = \alpha\delta = 0.$$

Since $\alpha \neq 0$, this requires $\delta = 0$. Otherwise, α, β, γ and δ must satisfy the inequality

$$\beta(\gamma - 2) \leqslant \alpha\delta \leqslant \beta(1+\gamma),$$

with $\beta > 0$.

In order to investigate whether the fluid tends to isotropy for all initial conditions, or not, write

$$g = x\sigma, \quad \text{where} \quad -\tfrac{1}{2} \leqslant x \leqslant 1,$$

and $$\delta = \beta f(1+n)(1+m)/2(1+r)(2+r), \quad \text{where} \quad \beta \geqslant 0, \quad -1 \leqslant f \leqslant 2.$$

Then, if $\dot{x} = 0$,

$$\sigma^2(1-x)(1+2x)\ddot{x} = xF(x)\dot{\sigma}^2, \tag{2.5}$$

where $F(x)$ is given in the appendix; from this it is clear that there are some fluids for which $F(x)$ can vanish for a value of x between $-\frac{1}{2}$ and 1. For such fluids it is possible to pose initial conditions for which there is a solution with a constant value of x. This shows that, for some fluids which satisfy all the requirements that have been made, there are initial conditions for which the solution does not tend to isotropy. For the rest of this section, therefore, we shall restrict attention to those fluids for which

$$F(x) < 0 \quad \text{for} \quad -\tfrac{1}{2} < x < 1.$$

Since the form of the equations for x and σ show that σ is not constant near the singular points and that σ decreases with time, x can be expressed as a function of σ by an equation of the form

$$\sigma^2(1-x)(1+2x)\,d^2x/d\sigma^2 = C(x, \sigma\, dx/d\sigma). \tag{2.6}$$

The function C, which is a cubic in its second argument, unless $\beta = 0$, is given in the appendix.

To investigate the behaviour of solutions of (2.6) near $x = 1$, write

$$x = 1 - \epsilon, \quad \tau = \log \sigma,$$

and approximate (2.6) to the lowest order in each term:

$$\epsilon\, d^2\epsilon/d\tau^2 = -C(1, -d\epsilon/d\tau).$$

Thispossesses solutions which vanish as τ decreases if

$$C(1, y) = 0$$

possesses any real neagative roots. Thus if the fluid is to tend to isotropy the equation may not have any such roots. Likewise

$$C(-\tfrac{1}{2}, y) = 0$$

may not possess any real positive roots. Unless these conditions are satisfied, some solutions of the original equation will tend asymptotically to the singular values of x.

When all the information gathered so far is used to express the constants a_1–a_9 in terms of the parameters m, n, r, f and β, the equation of the ν-fluid is found to be

$$a_3 T + \tfrac{1}{2} a_5 I \operatorname{tr}(T) = 0, \tag{2.7}$$

where the form of T is given in the appendix.

Since the equation must give a solution for isotropic decay it follows that a_5 cannot be chosen equal to $-\frac{2}{3}a_3$, and so the equation is equivalent to

$$T = 0. \tag{2.8}$$

The general conditions that energy should decay and that S should be positive definite can be investigated for (2.8) in a similar way to the special investigation of axisymmetric decay. Thus the zeros of the principal stresses must always be singular points. That this is in fact the case is most easily seen if the equation is regarded as an algebraic equation for the components of $\ddot{S}$ in a frame of reference whose axes coincide instantaneously with the principal axes of S. It can then be shown that the system of three equations which results for $\ddot{S}_{11}$, $\ddot{S}_{22}$ and $\ddot{S}_{33}$ is singular whenever a principal component of S vanishes.

In the same way energy decay can be investigated in the general case. Consider a set of initial conditions for which all the principal stresses are non-zero but the energy decay is arbitrarily small on a time scale defined in terms of $\operatorname{tr}(\dot{S}_0^2)$. Multiply (2.8) by S_0^{-1} and take the trace of the equation which results. With the condition $\dot{\sigma}_0$ arbitrarily small this is approximately

$$3\ddot{\sigma}_0 + \tfrac{3}{2}f\beta \operatorname{tr}(\dot{S}_0^2 S_0^{-1}) + \tfrac{1}{2}\beta(1-f)\operatorname{tr}(\dot{S}_0^2)\operatorname{tr}(S_0^{-1}) = 0.$$

The condition that $\dot{\sigma} < 0$ requires that $\ddot{\sigma}_0 \leqslant 0$ and so, for all initial conditions with $\dot{\sigma}_0$ arbitrarily small, the sum of the second two terms must be positive or zero. However, the ratio

$$3\operatorname{tr}(\dot{S}_0^2 S_0^{-1})/\operatorname{tr}(\dot{S}_0^2)\operatorname{tr}(S_0^{-1})$$

can then have any value in the open interval $(0, 2)$. The range of permissible values of f is therefore given by

$$-1 \leqslant f \leqslant 1.$$

It is now possible to see that for $\beta \neq 0$ these conditions, as well as being necessary for energy decay, are also sufficient, for suppose that, at some later value of t, $\dot{\sigma} = 0$. This would have to be a maximum value of the energy because of the sign of $\ddot{\sigma}$, in contradiction to the requirement that the energy was previously decreasing.

Finally, the complete set of conditions that have been found so far can be investigated to see if there are any solutions for m, r, n, β and f. The conditions that must be satisfied are:

$$m > r > -1, \quad n > r > -1, \quad \beta \geqslant 0, \quad |f| \leqslant 1; \tag{2.9}$$

$$F(x) < 0 \quad \text{for} \quad -\tfrac{1}{2} < x < 1; \tag{2.10}$$

$$C(1, y) \neq 0 \quad \text{for} \quad y < 0; \tag{2.11}$$

$$C(-\tfrac{1}{2}, y) \neq 0 \quad \text{for} \quad y > 0. \tag{2.12}$$

It is convenient to investigate the two cases $\beta = 0$ and $\beta > 0$ separately. In order to study the problem when $\beta \neq 0$ introduce the new parameters

$$\eta = 3(2+r)/4(m+n+3),$$
$$\xi = 2(m+n+3)(2+r)^2(1+r)/\beta(1-\tfrac{1}{2}f)(1+n)^2(1+m)^2,$$

and, when $f \neq -1$,

$$\zeta = 2(m+n+3)(2+r)^2(1+r)/\beta(1+f)(1+n)^2(1+m)^2.$$

The cubic $C(1,y)$ has two real turning points if

$$\xi^2 - \xi + 1 + 8\eta\xi > 0.$$

It can be shown that this is always so when $\eta > 0$ and $\xi > 0$. The coefficient of y in $C(1,y)$ is necessarily positive and the constant term is negative or zero, so that the cubic has at least one non-negative root. However,

$$\frac{27\xi(1+r)^2(2+r)}{(1+n)(1+m)(n+m+3)} C(1,y_1) = (\xi^2-\xi+1+8\eta\xi)^{\frac{3}{2}} + \xi^3 - \tfrac{3}{2}\xi^2 - \tfrac{3}{2}\xi + 1 - 12\xi\eta(2-\xi),$$

where y_1 is the lesser of the two values of y at which $C(1,y)$ turns. This is negative only if

$$(\tfrac{3}{2} - \tfrac{3}{2}\xi - 4\xi\eta)^2 (3\xi^2 + 32\xi\eta) < 0.$$

Since this condition cannot be satisfied it follows that all three roots are real, and since the coefficient of y^2 in $C(1,y)$ is strictly positive the sum of the three roots is negative. There is therefore at least one negative root. Consequently it is not possible to satisfy condition (2.11).

Condition (2.12) can be investigated in the same way when $f \neq -1$. It can then be shown that $C(-\frac{1}{2},y)$ has two turning points if

$$1 - \zeta + \zeta^2 + 2\eta\zeta > 0,$$

which is necessarily so. Since the coefficient of y^2 is negative, the greater value of y at which $C(-\frac{1}{2},y)$ turns, y_2, is necessarily positive and

$$\frac{27\zeta(1+r)^2(2+r)}{2(1+n)(1+m)(n+m+3)} C(-\tfrac{1}{2},y_2) = -1 + \tfrac{3}{2}\zeta + \tfrac{3}{2}\zeta^2 - \zeta^3 + 3\eta\zeta(2-\zeta) - (1-\zeta+\zeta^2+2\eta\zeta)^{\frac{3}{2}},$$

which cannot be positive. There is thus at least one positive zero of $C(-\frac{1}{2},y)$ when $f \neq -1$. Since the result is easily shown to be true when $f = -1$, it follows that condition (2.12) cannot be satisfied either.

The typical behaviour for axisymmetric relaxation of stress systems governed by (2.7) is shown in figure 1, in which $y = \sigma\dot{x}/\dot{\sigma}$ is plotted against the degree of anisotropy, x, for various initial conditions. It is a curious feature of such stress systems that, for some choices of the constants, the system can reach either a one-dimensional or a two-dimensional state in a finite time from certain initial conditions. When this happens the stress system either becomes stationary or, if it continues to change, it alters in a way that is not governed by (2.7). That such systems can be constructed is illustrated by the choice of f and β which makes

$$\zeta(1+\tfrac{2}{3}\eta) = \xi(1+\tfrac{8}{3}\eta) = 1.$$

(The conditions $\beta > 0$ and $|f| < 1$ are automatically satisfied if $r > -1$, and by choosing a sufficiently large value of m, condition (2.10) can also be satisfied.) It is then possible to show that for some initial values the stress becomes either one-dimensional or two-dimensional in a finite time, with non-zero energy.

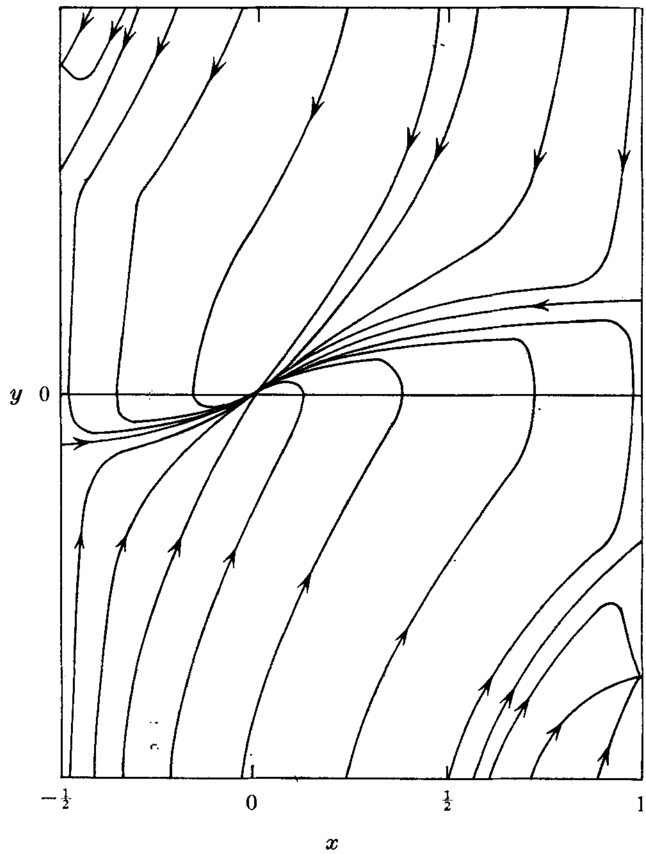

FIGURE 1. Typical solution curves of (2.7) in the form for axisymmetric stress given in (2.6); $y = \sigma\dot{x}/\dot{\sigma}$ is given as a function of the degree of anisotropy x. The direction in which the curves are traversed as t increases is shown by the arrows; σ decreases monotonically as t increases.

In those fluids for which $\beta = 0$, the function $C(x,y)$ is a quadratic in y, while $F(x)$ is a quadratic in x. If

$$F(x) \neq 0 \quad \text{at} \quad x = -\tfrac{1}{2} \quad \text{or} \quad 1,$$

it is simple to show that $C(1,y)$ and $C(-\frac{1}{2},y)$ each have two real zeros with opposite sign. Only if

$$r(n+m+3) = mn-2$$

do $F(-\frac{1}{2})$ and $F(1)$ both vanish. It can then be shown that the non-zero root of

$$C(-\tfrac{1}{2},y) = 0$$

is positive, and that the non-zero root of

$$C(1,y) = 0$$

is negative, so that neither of conditions (2.11) and (2.12) can be satisfied.

This concludes the demonstration that it is not possible to construct a doubly degenerate third-order ν-fluid of type I or type II which tends to the isotropic state for arbitrary initial conditions.

3. Fluids of type III

There are only two subdivisions of fluids of type III.

Type IIIa, for which $a_1 = a_2 = a_3 = 0$ and $a_4^2 + a_5^2 \neq 0$.

Type IIIb, for which $a_1 = a_3 = a_5 = 0$ and $a_2 \neq 0$.

All fluids of type IIIa which possess isotropic solutions that decay to zero in a manner governed by equation (1.1) also possess a more general solution of the form

$$S = S_0 + \tfrac{1}{3}I\left\{\frac{1}{(1+t/t_0)^{1+r}} - 1\right\}\operatorname{tr}(S_0),$$

where
$$r > -1, \quad t_0 = (1+r)\operatorname{tr}(S_0)/|\operatorname{tr}(\dot{S}_0)|.$$

It is therefore always possible to pose initial conditions appropriate to the equation for which S fails to remain positive definite, and so fluids of this type do not satisfy restriction 1. They will not be discussed further.

Only those fluids of type IIIb which possess isotropic solutions that decay and whose behaviour is governed by the equation will be considered; a necessary condition therefore is

$$a_2 + 3a_4 \neq 0.$$

The axisymmetric relaxation of stress in these fluids may be studied in the same way as in the last section.

The equations which govern g and σ are now

$$\sigma\ddot{\sigma} + \beta\dot{g}^2 - \frac{2+r}{1+r}\dot{\sigma}^2 = 0 \tag{3.1}$$

and
$$g\ddot{\sigma} - \frac{2+r}{1+n}\dot{g}\dot{\sigma} + \delta\dot{g}^2 = 0, \tag{3.2}$$

where n, r, β and δ are now defined by the ratios

$$\tfrac{1}{3}a_2 + a_4 : \tfrac{2}{9}a_7 + \tfrac{2}{3}a_9 : \tfrac{1}{3}a_6 + \tfrac{1}{9}a_7 + a_8 + \tfrac{1}{3}a_9 = 1 : \beta : -\frac{2+r}{1+r}$$

and
$$a_2 : a_6 + \tfrac{2}{3}a_7 : \tfrac{1}{3}a_7 = 1 : -\frac{2+r}{1+n} : \delta.$$

The interpretations of r and n are the same as in the previous section, so the power law for decay of isotropic stress and a tendency towards isotropy for small departures from it require, as before,

$$n > r > -1.$$

Equations (3.1) and (3.2) can be used to eliminate $\ddot{\sigma}$ and to obtain the equation

$$\left(x + \sigma\frac{dx}{d\sigma}\right)^2 (\beta x - \delta) + \frac{2+r}{1+n}\left(x + \sigma\frac{dx}{d\sigma}\right) - \frac{2+r}{1+r}x = 0 \tag{3.3}$$

for the degree of anisotropy x, equal to the ratio of g to σ. It follows that if

$$\beta x^2 - \delta x - (2+r)(n-r)/(1+r)(1+n) = 0 \quad \text{for} \quad -\tfrac{1}{2} < x < 1$$

there is an initial condition for which the fluid does not tend to isotropy.

If attention is now restricted only to those fluids for which

$$\beta x^2 - \delta x - (2+r)(n-r)/(1+r)(1+n) < 0 \quad \text{for} \quad -\tfrac{1}{2} < x < 1, \tag{3.4}$$

it is clear that (3.3) must possess at least one solution for y for each value of x in $(-\frac{1}{2}, 1)$, where

$$y = \sigma(dx/d\sigma);$$

if this is not so, the equations do not have solutions for all possible initial-value problems appropriate to their type. Further, at $x = -\frac{1}{2}$ equation (3.3) must have no positive roots, and at $x = 1$ it must have no negative roots. These two conditions are necessary since otherwise, provided that σ always decreases, it is possible to specify initial conditions which give solutions that cease to be positive definite.

If β and δ are not both zero, it follows that β must be strictly negative (from (3.3), (3.4) and the two conditions at $x = -\frac{1}{2}$ and 1), and

$$\beta \leqslant \min\{\delta, -2\delta\}.$$

When this is so, (3.3) has real roots if

$$(2+r)(1+r) \geqslant 4x(\delta - \beta x)(1+n)^2.$$

If the equality cannot be satisfied for any value of x in $(-\frac{1}{2}, 1)$, there are initial-value problems whose solutions cannot tend to isotropy and also remain positive definite, since only one branch of (3.3) can pass through $x = y = 0$, and the two branches do not intersect. Equality occurs if

$$x^2 - \frac{\delta}{\beta}x + \frac{(2+r)(1+r)}{4\beta(1+n)^2} = 0. \tag{3.5}$$

Unless this equation has a double root the condition that (3.3) has real roots is violated for some acceptable values of x; since the last term is necessarily negative it is not possible to satisfy the condition. The only circumstance under which this fails to be true is when the roots of (3.5) are $-\frac{1}{2}$ and 1; even then, however, there are solutions which do not tend to isotropy as σ decreases since the double root of (3.3) for y cannot be zero at both points. This is apparent since the branches of the solution curves for y as a function of x do not intersect or touch except at $x = -\frac{1}{2}$ and 1; from these points, one curve in each case tends to infinity (at $x = \frac{1}{2}$), and since the solutions of the equation move away from them as σ decreases, there is one solution that does not tend to isotropy.

Thus if all solutions of (3.1) and (3.2) are to tend to isotropy and to remain positive definite, β and δ must both be zero. When this is so,

$$\sigma = \sigma_0/(1+t/t_0)^{1+r}, \quad g = g_0/(1+t/t_0)^{1+n},$$

where σ_0, g_0 and t_0 are constants of integration. This solution has all the properties required of it provided only that

$$n > r > -1. \tag{3.6}$$

The fluid from which these values of β and δ arise has the equation

$$a_2 T + a_4 \operatorname{tr}(T) I = 0, \tag{3.7}$$

where

$$a_2 + 3a_4 \neq 0$$

and

$$T = \ddot{\sigma} S - \frac{1}{3}\frac{2+r}{1+r}\dot{\sigma}^2 I - \frac{2+r}{1+n}\dot{\sigma}(\dot{S} - \tfrac{1}{3}\dot{\sigma} I). \tag{3.8}$$

The exact and unique solution of (3.7) for an initial-value problem at $t = 0$ is

$$S = \frac{S_0 - \frac{1}{3}\sigma_0 I}{(1+t/t_0)^{1+n}} + \frac{\frac{1}{3}\sigma_0 I}{(1+t/t_0)^{1+r}}, \tag{3.9}$$

where S_0 and σ_0 are the initial values of S and σ at $t = 0$ and

$$t_0 = (1+r)\,\sigma_0/|\dot{\sigma}_0|,$$

where $\dot{\sigma}_0 < 0$ is the initial value of $\dot{\sigma}$. It is simple to show that S tends to isotropy and that if S is initially positive semi-definite it becomes positive definite and remains so if and only if condition (3.6) holds.

There is therefore a class of fluids of type III for which all solutions remain positive definite and tend to isotropy, and for which the energy decreases monotonically to zero. This fluid has only two non-zero constants with significance in the problem considered and these two must satisfy (3.6). The equation of the fluid is given in (3.7).

4. Discussion

We have seen that the only fluids whose stress is governed by (1.1) and whose stress remains positive definite and tends towards isotropy, and whose energy decays, are of type III. Their equation is

$$a_2\left\{\ddot{\sigma} S - \frac{1}{3}\frac{2+r}{1+r}\dot{\sigma}^2 I - \frac{2+r}{1+n}\dot{\sigma}(\dot{S} - \tfrac{1}{3}\dot{\sigma} I)\right\} + a_4\left(\sigma\ddot{\sigma} - \frac{2+r}{1+r}\dot{\sigma}^2\right) I = 0,$$

where $a_2 \neq 0$, $a_2 + 3a_4 \neq 0$ and $n > r > -1$.

The general solution of the equation is given in (3.9), and has all the properties of homogeneous turbulence listed as (i)–(iv) in § 1. This family of fluids can therefore be used to model turbulence at infinite Reynolds number, and it is the only family of doubly degenerate third-order ν-fluids that can be considered for the purpose. It is therefore instructive to compare the consequences of the solution (3.9) with the results of experiments and to attempt to estimate values for n and r.

For the decay of isotropic turbulence, Batchelor & Townsend (1948) found that in an initial period the energy decayed approximately inversely with t, and this corresponds to a value for r of zero. A summary of the results of these and later experiments for isotropic and slightly anisotropic turbulence is given by Comte-Bellot & Corrsin (1966). Most of the experiments listed by them are for grid Reynolds numbers in the range 5000–150 000, and they deduce a typical value for r of 0·25.

Townsend (1951) noted that anisotropic turbulence produced by a gauze tended to return to isotropy, although very slowly. He confirmed this tendency

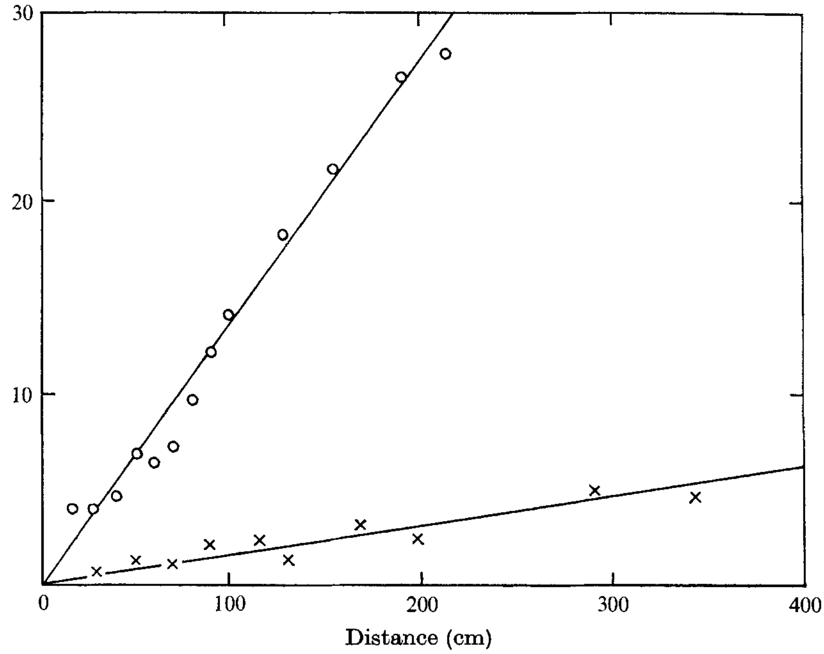

FIGURE 2. The observations of Tucker & Reynolds (open circles) for K_1^{-3} plotted against distance from the end of the distorting section of the wind tunnel and the observations of Comte-Bellot & Corrsin (crosses) for $x^{-3} \times 10^{-4}$ plotted against distance from the turbulence-producing grid.

(Townsend 1954) with a higher initial level of anisotropy produced by a uniform distortion in a specially shaped wind tunnel. Somewhat similar experiments were performed by Tucker & Reynolds (1968), and these can be used to estimate $n-r$. Figure 2 shows the relation of the number

$$K_1 = \frac{\overline{u_3^2} - \overline{u_1^2}}{\overline{u_3^2} + \overline{u_1^2}}$$

to the distance down the wind tunnel, where $\overline{u_1^2}$ and $\overline{u_3^2}$ are the mean-square turbulent velocity components perpendicular to the axis of the tunnel. In these experiments the mean-square turbulent velocity component parallel to the axis was approximately equal to $\frac{1}{2}(\overline{u_1^2} + \overline{u_3^2})$. The solution (3.9) predicts that K_1 should be proportional to the $-(n-r)$th power of distance down the tunnel from a suitably chosen origin under these circumstances. In figure 2, K_1^{-3} is plotted against distance; the relation between the observational points is described well by a straight line, giving

$$n - r \simeq 0{\cdot}33.$$

Tucker & Reynolds compare their experiments with those of Uberoi (1963), and conclude that the value of r appropriate to their experiments is 0·2.

The experiments of Comte-Bellot & Corrsin for axisymmetric decay of turbulence are consistent with this value of $n-r$, and figure 2 also shows their observations for x^{-3} plotted against distance, where x is the degree of anisotropy defined as in §§ 2 and 3 of this paper.

In conclusion it seems that the behaviour predicted from the modelling of homogeneous turbulence of infinite Reynolds number by a doubly degenerate third-order ν-fluid is consistent with the known experimental results. The simple way in which the structure of the stress tensor separates into the isotropic part and the deviatoric stress is remarkable. It suggests that results of experiments on anisotropic homogeneous turbulence might profitably be presented in this manner, rather than by attempting to describe the principal components separately.

From the study of (1.1), which possesses nine constants whose values are not initially known, we have been able to show that it cannot model various very simple properties of homogeneous turbulence unless five of these constants are exactly determined while two others have to satisfy inequality constraints. That it is possible to find such a high proportion of the constants by such simple arguments is not obvious beforehand, but that it can be done in one type of problem suggests that it may also be possible in others. If this is indeed so, it seems that the modelling of turbulence by means of a ν-fluid could become a method with considerable practical value.

I should like to thank Professor Ian Proudman and Dr Ian Cook for many interesting and valuable discussions on the behaviour of ν-fluids and the philosophy of their use for the description of turbulence.

Appendix

(i) The function F which appears in (2.5) has the form

$$F(x) \equiv \beta x^3(1-f)+x^2\left\{\frac{2(r+2)(n+m+3)}{(n+1)(m+1)}+\frac{(n+1)(m+1)}{(r+1)(r+2)}\beta+\beta\right\}$$
$$-x\left\{\tfrac{1}{2}\beta f\frac{(n+1)(m+1)}{(r+1)(r+2)}+\frac{r+2}{r+1}\right\}-\frac{(n-r)(m-r)}{(r+1)^2}.$$

(ii) The function C which appears in (2.6) has the form

$$C(x,y) \equiv xF(x)+y\left\{\frac{(n+1)(m+1)}{(r+1)(r+2)}\beta x(2x-f)+\frac{n+m+3}{r+1}-\frac{r+2}{r+1}(1+x)\right.$$
$$\left.+\frac{4(r+2)(n+m+3)}{(n+1)(m+1)}x^2-2(1+x-2x^2)-3\beta x^2(fx-1-x)\right\}$$
$$+y^2\left\{\beta\frac{(1+n)(1+m)}{(r+1)(r+2)}(x-\tfrac{1}{2}f)-3\beta x(fx-1-x)+2x\frac{(r+2)(n+m+3)}{(n+1)(m+1)}\right\}$$
$$+\beta y^3(1+x-fx).$$

(iii) The function T is (2.7) is given by

$$T \equiv \ddot{\sigma}S-\frac{1}{3}\frac{2+r}{1+r}\dot{\sigma}^2I+\frac{3}{2}\frac{(r+1)(r+2)}{(n+1)(m+1)}(S\ddot{S}+\ddot{S}S-\tfrac{2}{3}\ddot{\sigma}S)$$
$$-\frac{(r+2)(n+m+3)}{(n+1)(m+1)}\dot{\sigma}(\dot{S}-\tfrac{1}{3}\dot{\sigma}I)+\tfrac{3}{2}f\beta(\dot{S}-\tfrac{1}{3}\dot{\sigma}I)^2$$
$$+\tfrac{1}{2}\beta(1-f)\,I\{\operatorname{tr}(\dot{S}^2)-\tfrac{1}{3}\dot{\sigma}^2\}.$$

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